ELEN 4810 Midterm Exam

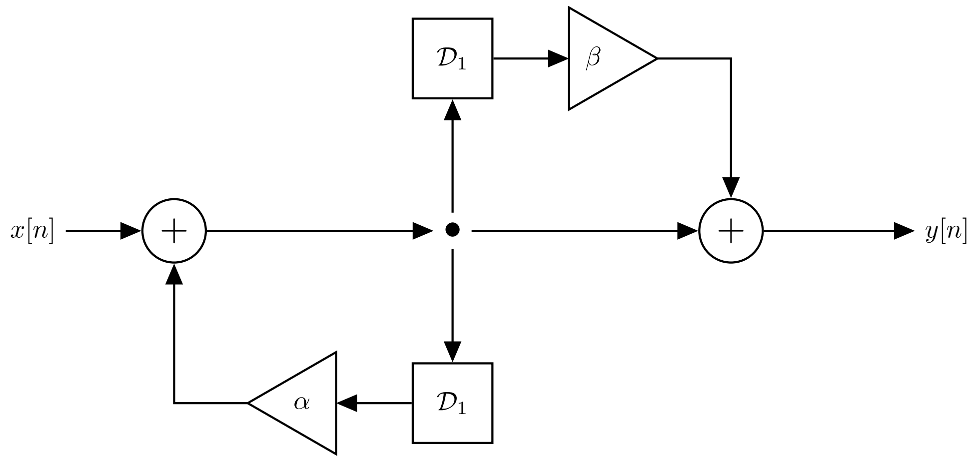
Monday, October 30, 2023, 1:10-3:10 PM. One sheet of handwritten notes is allowed. No electronics of any kind are allowed. Please record your answers in the exam booklet. Raise your hand if you need additional scratch paper.

There are a total of 4 questions. Good luck!

Name:

Uni:

1. Systems in Time and Frequency. Consider the causal linear, time invariant system corresponding to the following block diagram:



Here, $\mathcal{D}_1$ denotes an ideal delay by one sample, and the triangular blocks denote multiplication by complex scalars α and β, respectively.

Please answer the following questions:

Part (a). For what α and β is the system *stable*?

[Note: for full credit, please specify all possible values of (α, β)]

Part (b). What is the *frequency response* $H(e^{j\omega})$ of the system?

[Note: here, you only need to consider values of (α, β) for which the system is stable]

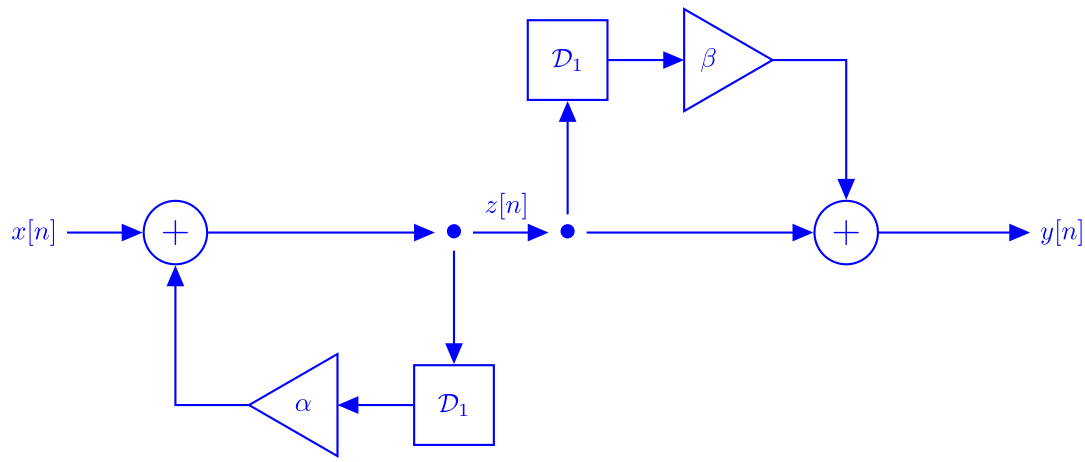
Part (c). What is the *impulse response* $h[n]$ of the system? Is this an FIR or IIR system?

Part (d). Consider the constant input $x[n] = 1$. *For what choices of* (α, β) *is the output* $y[n] = 0$ *zero for all* n*?* Please make your answer as broad as possible, or if no such (α, β) exist, explain why.

Part (e). Consider the input $x[n] = 1 + 3(-1)^n$. *For what choices of* (α, β) *is the output* $y[n] = 0$ *zero for all* n*?* Again, please make your answer as broad as possible, or if no such (α, β) exist, explain why.

Answer to Problem 1:

Part (a). [2 points] This system is a composition of two systems:



The first is an IIR system, which is stable if and only if $|\alpha| < 1$; the second is an FIR system, which is stable for all choices of β:

$$\boxed{\text{All } \beta \in \mathbb{C},\ |\alpha| < 1.}$$

Part (b). [2 points] We have

$$z[n] = x[n] + \alpha z[n-1].$$

Taking Fourier transforms, we have

$$\left(1 - \alpha e^{-j\omega}\right) Z(e^{j\omega}) = X(e^{j\omega}).$$

Simlarly,

$$Y(e^{j\omega}) = \left(1 + \beta e^{-j\omega}\right) Z(e^{j\omega}) = \frac{1 + \beta e^{-j\omega}}{1 - \alpha e^{-j\omega}} X(e^{j\omega}), \tag{1}$$

and so

$$\boxed{H(e^{j\omega}) = \tfrac{1+\beta e^{-j\omega}}{1-\alpha e^{-j\omega}}.}$$

Part (c). [2 points] We take the inverse Fourier transform of H. Notice that

$$\alpha^n u[n] \overset{\text{DTFT}}{\to} \frac{1}{1 - \alpha e^{-j\omega}}.$$

Using this and the shifting property of the DTFT, we have that

$$\alpha^n u[n] + \beta \alpha^{n-1} u[n-1] \to \frac{1 + \beta e^{-j\omega}}{1 - \alpha e^{-j\omega}},$$

giving

$$\boxed{h[n] = \alpha^n u[n] + \beta \alpha^{n-1} u[n-1].}$$

Part (d). [2 points] The system output is zero if and only if $H(e^{j0}) = 0$, i.e.,

$$\frac{1+\beta}{1-\alpha} = 0,$$

which is true if and only if $\beta = -1$:

$$\boxed{\beta = -1, \text{ all } \alpha \in \mathbb{C}.}$$

Part (e). [2 points] The system output is zero if and only if $H(e^{j0}) = 0$ *and* $H(e^{j\pi}) = 0$, i.e.,

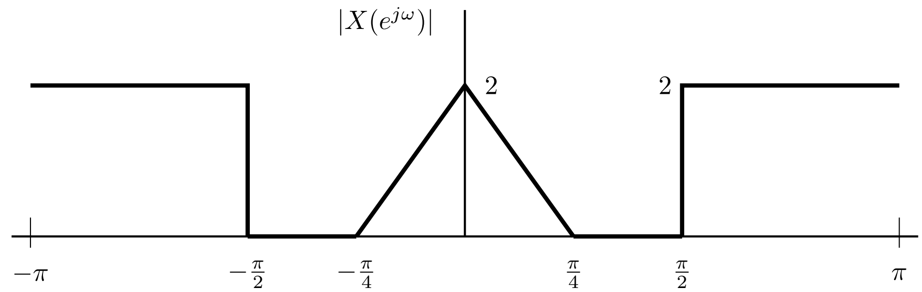
$$\frac{1+\beta}{1-\alpha} = 0, \quad \text{and} \quad \frac{1-\beta}{1+\alpha} = 0.$$

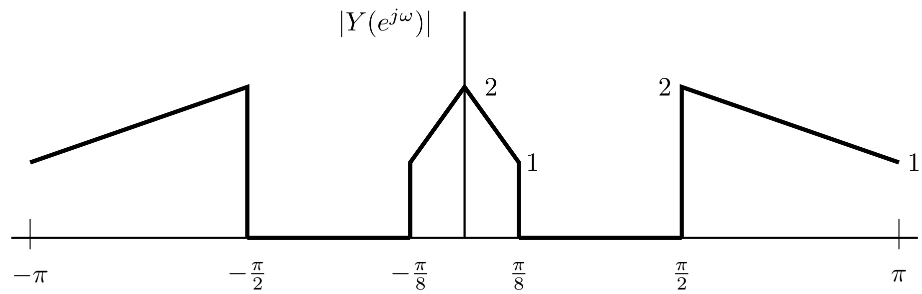
This forces $\beta = -1$ *and* $\beta = 1$, which is impossible.

$$\boxed{\text{There is no choice of } \alpha, \beta \text{ for which the output is zero.}}$$

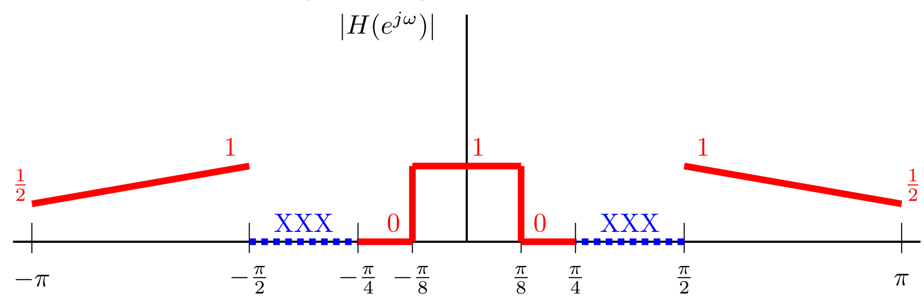
2. LTI Systems in Frequency Domain. A signal $x[n]$ is input to an LTI system with impulse response $h[n]$, producing output $y[n]$. Below, we plot magnitude of $X(e^{j\omega})$ and $Y(e^{j\omega})$.

Part (a). *Please plot the magnitude $H(e^{j\omega})$. Please label your graph as clearly as possible, and indicate any points for the magnitude response cannot be determined:*





Your answer to Part (a) here: [4 points]



XXX – cannot determine

Part (b). For which of the following signals $x[n]$ is it possible to determine the impulse response $h[n]$ from the convolution $y[n] = x[n] * h[n]$? Please briefly explain your answers:

- 1. $x[n] = \frac{\sin\left((n-1)\pi/5\right)}{(n-1)\pi/5}$.
- 2. $x[n] = \cos(n\pi/4)$

- 3. $x[n] = \delta[n-10]$
- 4. $x[n] = (1/4)^n u[n]$

Answer to Problem 2: **Part (b). [4 points]** It is possible to determine $h[n]$ from $x[n]$ and $y[n]$ if and only if

$$|X(e^{j\omega})| > 0$$

for all ω; in this case, we have

$$H(e^{j\omega}) = Y(e^{j\omega})/X(e^{j\omega}),$$

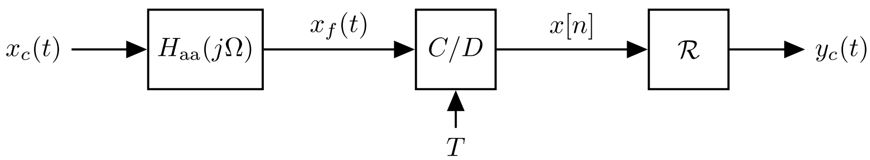
and

$$h[n] = \text{DTFT}^{-1}\Bigg(H(e^{j\omega})\Bigg).$$

We have $|X(e^{j\omega})| > 0$ for all ω for signals 3 and 4 (whose modulus Fourier transforms are $|X_3(e^{j\omega})| = 1\ \forall\omega$ and $|X_4(e^{j\omega})| = |1/(1-\frac{1}{4}e^{-j\omega})| > 0\ \forall\omega$, and not for 1 and 2 (whose Fourier transforms are a box and a pair of spikes, which vanish for certain frequencies ω).

$$\boxed{\text{We can determine } h \text{ for signals 3 and 4, but not for signals 1 and 2.}}$$

3. Sampling and Aliasing. Consider the following block diagram:



Here,

- $x_c(t)$ is a continuous time signal
- $H_{\text{aa}}(j\Omega)$ is a continuous time filter whose frequency response will be specified below
- $x_f(t)$ is a continuous-time signal
- The box marked $\mathcal{R}$ will be specifed below.

Please answer the following questions:

Part (a). Suppose that $\mathcal{R}$ is an ideal discrete to continuous converter, with period T seconds. We wish to ensure that $y_c(t) = x_c(t)$ for the broadest possible set of inputs $x_c(t)$.

How should we choose $H_{\text{aa}}(j\Omega)$?

[Note: please specify H_{aa} as a function of Ω. Your answer should depend on T]

Part (b). *With your choice of H_{aa} from part (a), for which inputs x_c can we guarantee that $y_c(t) = x_f(t)$?*

[Note: Please make your answer as broad as possible for full credit. Please notice that Part (b) of the question asks whether $y_c = x_f$, *not* whether $y_c = x_c$!]

Part (c). Now suppose we are interested in reconstructing ***bandlimited*** signals $x_c(t)$ satisfying

$$X_c(j\Omega) = 0, \quad |\Omega| \geq \Omega_M. \tag{2}$$

Suppose further that we know that the input signal $x_c(t)$ is ***real valued***. We can use this property to reduce the sampling rate, by making different choices of the filter H_{aa} and the block $\mathcal{R}$.

Please specify a filter $H_{aa}(j\Omega)$, a sampling period T, and describe a reconstruction procedure $\mathcal{R}$ which ensures $y_c(t) = x_c(t)$ for all such bandlimited, real-valued $x_c(t)$. For full credit, please make T as large as possible!

[Hint: You may find it helpful to note that for a real valued signal $x_c(t)$, the Fourier transform is conjugate symmetric: $X_c(j\Omega) = X_c(-j\Omega)^*$]

Answer to Problem 3:

Part (a). [3 points] We should choose H_{aa} to be an ideal anti-aliasing filter with cutoff $\Omega_s/2 = \pi/T$:

$$\boxed{H_{\text{aa}}(j\Omega) = \begin{cases} 1 & |\Omega| \le \frac{\pi}{T} \\ 0 & \text{else.} \end{cases}}$$

Part (b). [3 points] The signal $x_f(t)$ is *always* bandlimited with bandlimit $\Omega_s/2$, and so

$$\boxed{y_c(t) = x_f(t) \textit{ for all } \text{inputs } x_c(t).}$$

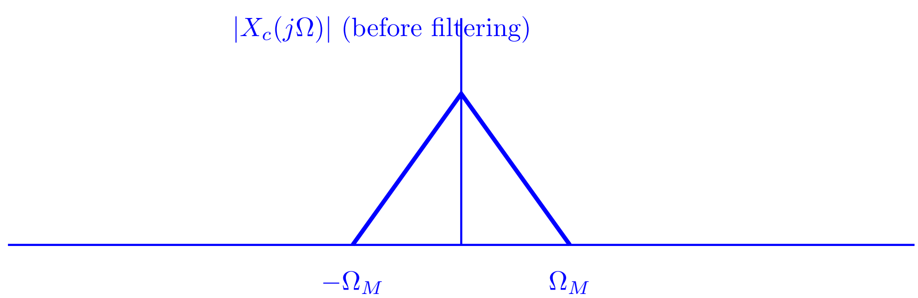
Part (c). [3 points] Standard Shannon-Nyquist reconstruction requires $\Omega_s \ge 2\Omega_M$, i.e., $T \le \pi/\Omega_M$.

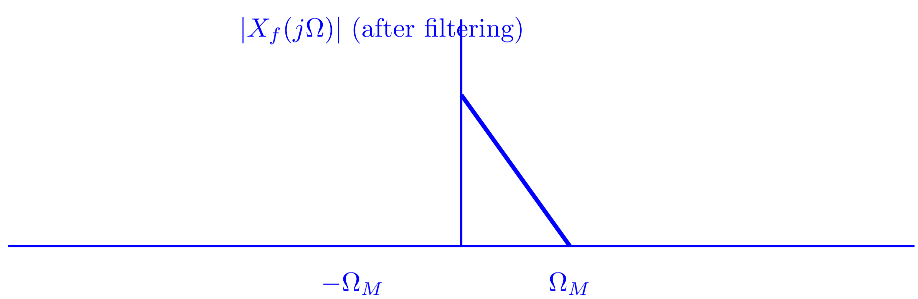
For real-valued signals $x_c(t)$, the fourier transform is conjugate symmetric, i.e., $X_c(j\Omega) = X_c^*(-j\Omega)$, and so $x_c(t)$ can be reconstructed from knowledge of $X_c(j\Omega)$ for $0 \le \Omega \le \Omega_M$.

In particular, we can set

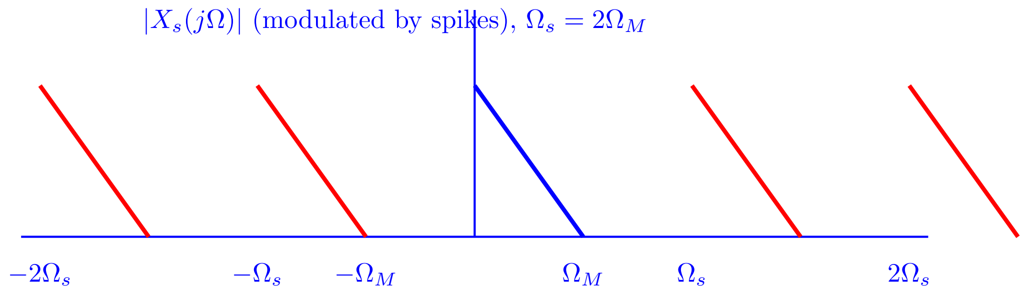
$$H_{\text{aa}}(j\Omega) = \begin{cases} \frac{1}{2} & \Omega = 0, \\ 0 & 0 < \Omega \le \Omega_M, \\ 0 & \text{else.} \end{cases}$$

With this choice, $X_f(j\Omega)$ has nonzero frequency content only for $0 \le \Omega < \Omega_M$:

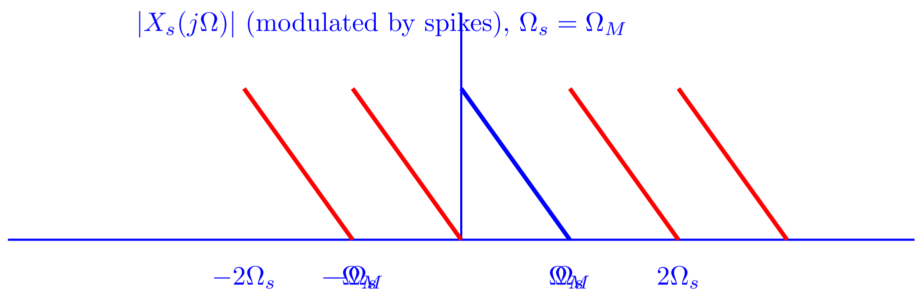




Consider an ideal C/D converter with sampling rate Ω_s (and sampling period $2\pi/\Omega_s$). This converter begins by modulating $x_f(t)$ with a spike train $s(t)$, producing



The above picture corresponds to Nyquist-rate sampling, i.e., $\Omega_s = 2\Omega_M$. Using the fact that X_f is only nonzero between 0 and Ω_M, we can reduce the sampling rate by a factor of 2 without causing overlap in the interval $0 \le \Omega < \Omega_M$:



We can recover $X_c(j\Omega)$ from $X_s(j\Omega)$ by (i) applying a reconstruction filter of the form

$$H_r(j\Omega) = \begin{cases} T & 0 \le \Omega \le \Omega_M, \\ 0 & \text{else,} \end{cases}$$

to produce a time domain signal $\widehat{x}(t)$, and then recovering the frequency content from $-\Omega_M < \Omega < 0$ by conjugate symmetry. Formally

$$y_c(t) = \widehat{x}(t) + \widehat{x}^*(t),$$

we obtain $y_c(t) = x_c(t)$.

The maximum sampling period is $T = 2\pi/\Omega_M$, i.e., twice the maximum sampling period for Nyquist-rate sampling.

Note: full credit to answers which (i) use conjugate symmetry by choosing an antialiasing filter that passes only $0 \le \Omega \le \Omega_M$, and (ii) achieve the correct maximal sampling period.

4. Discrete Fourier Transform and Convolution with a Filterbank. Consider real-valued discrete-time signal $x[n]$, which satisfies

$$\begin{cases} x[n] > 0 & n = 0, 1, \ldots, 127, \\ x[n] = 0. & \text{else} \end{cases} \tag{3}$$

We are given a collection of filters $h_1[n], \ldots, h_7[n]$ satisfying

$$h_i[n] = \begin{cases} h_i[n] = \cos\left(\frac{2\pi n}{2^i - 1}\right) & 0 \le n \le 2^i - 1. \\ 0 & \text{else} \end{cases} \tag{4}$$

We wish to compute the (linear) convolutions $y_1 = x * h_1, y_2 = x * h_2, \ldots, y_7 = x * h_7$ of x with each of the filters h_i.

Please consider the following proposed procedure: we compute the discrete fourier transforms

$$\begin{aligned} X &= \mathrm{DFT}_N\{x\}, \\ H_i &= \mathrm{DFT}_N\{h_i\}, \qquad i = 1, \ldots, 7. \end{aligned}$$

and set

$$\widehat{y}_i[n] = \mathrm{DFT}_N^{-1}\{H_i[k]X[k]\}, \qquad i = 1, \ldots, 7.$$

Please answer the following questions about this procedure:

Part 1. Suppose we set $N = 172$. *Which of the convolutions is computed correctly?* I.e., for which values of i is $\widehat{y}_i[n] = y_i[n]$ for all n?

Part 2. *What is the smallest N for which* **all** *of the convolutions are computed correctly?* I.e., for which N is $\widehat{y}_i[n] = y_i[n]$ for all n and all $i = 1, \ldots, 7$?

Part 3. Suppose again that $N = 172$, but now that we only care about values of n satisfying $8 \le n \le 120$. For which i is it true that

$$\widehat{y}_i[n] = y_i[n] \quad n = 8, \ldots, 120 \quad ? \tag{5}$$

Answer to Problem 4:

We begin by noticing that x has length 128, and h_i has length 2^i. The length of y_i are $128 + 2^i - 1$. In particular,

- y_1 has length $128 + 2 - 1 = 129$
- y_2 has length $128 + 4 - 1 = 131$
- y_3 has length $128 + 8 - 1 = 135$
- y_4 has length $128 + 16 - 1 = 143$
- y_5 has length $128 + 32 - 1 = 159$
- y_6 has length $128 + 64 - 1 = 191$
- y_7 has length $128 + 128 - 1 = 255$

Part 1. [3 points] Any convolutions for which $N \geq \text{length}(y_i)$ are correctly computed:

$$\boxed{\text{With } N = 172\text{, we correctly compute } y_1, \ldots, y_5.}$$

Part 2. [3 points] We need to zero pad to at least output length, so

$$\boxed{\text{All convolutions are correctly computed when } N \geq 255.}$$

Part 3. [3 points] $y_1, \ldots, y_5$ are still computed correctly. In $\widehat{y}_6$, the first $191 - 172 = 19$ entries are still not computed correctly. Hence, we do not correctly compute $\widehat{y}_6$ on the desired entries.

$$\boxed{\text{We still correctly compute } y_1, \ldots, y_5\text{, but not } y_6 \text{ and } y_7.}$$

Scratch paper: